On the curve $y = \left\{\frac{1}{x^2+\sin^2\psi}\right\}^{\frac{3}{2}}$, *and its connection with an Astronomical Problem.* By Mrs. W. H. Young (Miss Grace Chisholm).

(*Communicated by Professor G. H. Darwin.*)

Introduction. The Problem and its Origin.

The problem of the determination of the orbit of a comet (or other heavenly body) from three observations leads to the study of the following equation :—

$$m - z = \frac{m}{(1-2z\cos\psi+z^2)^{\frac{3}{2}}} = \frac{m}{\lambda^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here m and ψ are certain constants calculated from the observations ; ψ, being the apparent angular distance between the comet and the Sun, lies between 0° and 180° ; m may have any value, but the sign of m is positive or negative according as the apparent path of the comet is convex or concave toward the Sun.*

The unknown quantity z is the ratio of the geocentric distances of the comet and Sun at the time of the mean observation. Hence we perceive that we wish to determine for z a real positive value. The quantity λ, which is obviously always real, is likewise to be taken positive, for it represents the ratio of the heliocentric distances of the comet and Earth at the time of the mean observation.

If we rationalise the equation by squaring both sides we obtain an equation of the eighth degree in z, which, as such, possesses eight complex solutions, conjugate in pairs. One of these is, however, obviously zero ; hence at least one of the other seven solutions is real. The solution $z = 0$ corresponds to the case where we have been observing, not a heavenly body at all, but a speck in our own atmosphere, or even in the telescope itself ; a mistake which is, theoretically at least, always possible.

We have, then, seven solutions to take into account, of which an odd number ($> 0 \overline{\overline{<}} 7$) are real. Possibly no one of these real solutions is positive ; in this case the calculations of m and ψ from the observations, or ultimately the observations themselves,

* $m = \frac{\tau_{\prime}\tau_{\prime\prime\prime}}{2} \frac{\sin P_{\prime\prime}}{\sin p_{\prime\prime}} \frac{1}{R_{\prime\prime}{}^3}$,

$P_{\prime\prime}$ and $p_{\prime\prime}$ being the perpendiculars from the Sun and comet respectively at the time of the mean observation on the great circle through the extreme positions of the comet ; $R_{\prime\prime}$ is the distance at the same time of the Earth and Sun ; $\tau_{\prime}$ is the interval of time between the first and second observations multiplied by Gauss's constant ; and $\tau_{\prime\prime\prime}$ is the same quantity for the second and third observations (Oppolzer, p. 362).

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are incorrect. But if one single real root be positive then it gives the complete solution of the astronomical problem, and the orbit can be determined. Finally, if more than one real positive solution exists, the observations are insufficient, and the comet must be observed again to discriminate which of the roots is the proper one to take.

We see that the discussion of the roots of the equation (1) has a special interest for astronomers. Now, since the problem as it stands presents difficulties to purely analytical investigation, it was early suggested that it should be treated by geometry, and various geometrical methods have been employed for this purpose. The complete geometrical discussion has not, however, I believe, been given, and I am not aware that the problem has been discussed afresh since the publication of Oppolzer's *Elemente der Bahnbestimmung* (1882). In this last book the geometry is dismissed very shortly, nor is a complete analytical discussion given; and there is one absolute error as to the limits of solution the detection of which led to the writing of this paper.

The geometrical method which I use differs slightly from Oppolzer's and from that used by any of his predecessors. I have added a note on kindred methods to mine, and the results already obtained by them.

I have only to add that I believe I may offer this as a complete geometrical discussion, and that I have also been enabled in this way to give a simple analytical formula as a criterion for the existence of two roots, corresponding to the old formula for the existence of one root only.

I have added a note on Lambert's curvature theorem and its connection with the geometry.

§ 1. *Notation.*

I shall in this paper use the abbreviations s, c for $\sin\psi$ and $\cos\psi$; s is then always positive, and c is positive or negative according as ψ is less or greater than 90°. We have, however, in all cases

$$s^2+c^2=1, \qquad (2)$$

I then put

$$x=z-c, \qquad (3)$$

and replace the equation (1) by the two equations

$$y=\frac{1}{(x^2+s^2)^{\frac{3}{2}}}=\frac{m}{\lambda^3}, \qquad (4)$$

$$my=m-c-x. \qquad (5)$$

§ 2. *Geometry of the Problem.*

Let us now regard x and y as ordinary Cartesian coordinates, the axis of x being horizontal and that of y vertical. The equation (4) then represents a certain plane curve, the equation (5)

a straight line whose intersections with the curve are the object of investigation.

§ 3. *The Astronomical Point, Critical Point, and Critical Line.*

The point of intersection of the straight line (5) with the curve (4) I call, for obvious reasons, the astronomical point.

The line (5) meets the curve once at the point $(-c, 1)$, which I call the critical point. The vertical line through the critical point I call the critical line: its equation is

$$x + c = 0,$$

or

$$z = 0.$$

The astronomical point, being one for which z is positive, lies to the right of the critical line. Hence we need only consider such intersections of the line (5) with the curve (4) as lie to the right of the critical point.

From the form of the equation we see that $(-m)$ being the cotangent of the inclination of the line to the horizontal axis, m is the distance from the critical line of the point where the line (5) meets the horizontal axis, and is measured to the right if m be positive, to the left if m be negative. This gives a very simple construction for the line (5):

Find the critical point C and draw the vertical line through it. Measure a distance, m (to the right if m be positive, to the left if m be negative), along the horizonal axis, and join the point so found to C: this line is the line (5).

§ 4. *The Geometry of the Curve.*

Remembering that λ is to be a positive quantity, we see that the curve

$$y = \frac{1}{\lambda^3} = \frac{1}{(x^2 + s^2)^{\frac{3}{2}}}$$

lies entirely above the horizontal axis. It is, in fact, only one branch of the algebraic curve of the eighth degree, whose equation is got by rationalising the above equation: this curve possesses a second branch, the reflexion of the first in the horizontal axis, analytically to be distinguished from the upper branch by the fact that in the lower branch λ is always negative. With this lower branch, however, we are not concerned.

Seeing that

$$\frac{dy}{dx} = -3\lambda^{-5}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we recognise that the horizontal axis is an asymptote to our curve, and that there is one other horizontal tangent, at the point M in the figure, where the curve meets the axis of symmetry $(x = 0)$. There are no vertical tangents, the steepest inclination being at the inflexions, determined by the equation

$$0 = \frac{d^2y}{dx^2} = 3\lambda^{-7}(4x^2 - s^2),$$

that is to say,

$$x = \pm \frac{s}{2}, \quad \ldots \ldots \ldots \quad (7)$$

this determines the two inflexions V, W.

Substituting for x in (6), we get the tangent of the angle of inclination of the inflexional tangents given by the expression

$$\frac{3}{2}\left(\frac{4}{5}\right)^{\frac{5}{2}} \frac{1}{s^4}.$$

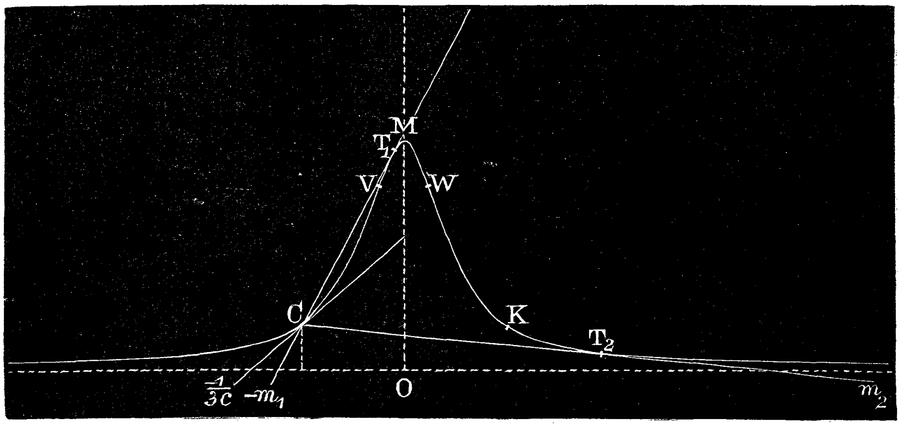

Fig. 1 ($0 < \psi < \tan^{-1}2$).

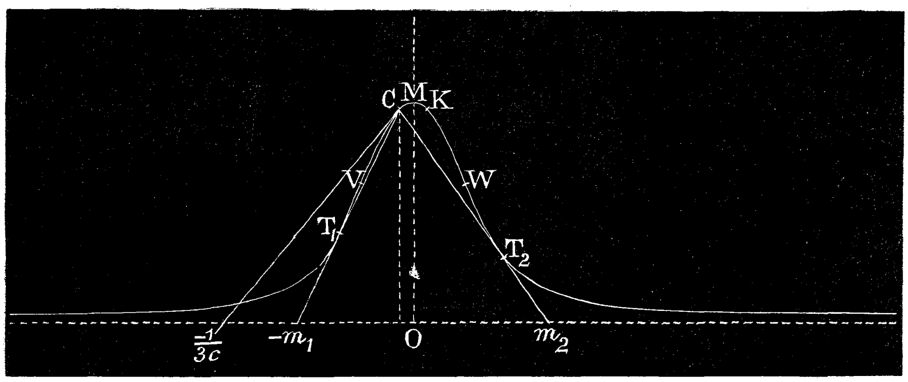

Fig. 2 ($\tan^{-1}2 < \psi < 90°$).

Thus the nearer ψ is to a right angle, the flatter is the curve (see figures). The form of the curve varies otherwise very little with ψ. The only thing which alters materially with ψ is the position of the critical point with respect to the two inflexions. From 0° to 63° 26′ 8″ ($\tan^{-1} 2$,) the critical point lies to the left of both inflexions, as in fig. 1. When $\psi = 63° \ 26' \ 8''$ we have

$$c - \frac{1}{2} s = 0,$$

and the critical point coincides with V. Between this value and 116° 33′ 52″ the critical point lies between the inflexions.

Lastly from 116° 33′ 52″ to 180° the critical point lies to the right of both inflexions.

Now inspection of the figures shows us at once, when the curve has been drawn and the critical point C marked on it, how the number of intersections of the line (5) with the curve which lie to the right of C are determined by the inclination of the line, that is, by the value of m. We see at once that there are three important values of m, corresponding to the tangent *at* the critical point, and the two tangents *from* the critical point, these separate values of m, for which there are respectively no solutions, one solution, two solutions of the problem. The first result got by inspection of our figure is the following: *There cannot be more than two solutions of the astronomical problem.*

For it is obvious that the line can in no case intersect the curve in more than three points, one of which is the critical point itself.

§ 5. *Condition for one Solution only.*

Consider first $0° < \psi < 90°$. The figures then show that when m is negative and sufficiently great, there is a single intersection to the right of C. This goes on until the line touches the curve at the critical point. For this line we have

$$m=-\frac{1}{3c},$$

as may be seen by substituting the coordinates of the critical point in the equation (6).

Hence there is one, and only one, solution if, and only if, m be negative and greater in absolute value than $-\frac{1}{3c}$.

If $90° < \psi < 180°$, the critical point lies to the right of the axis of symmetry, and it is obvious that the same conditions hold if we change the sign of m. Both cases are included if we state the condition for the existence of one, and only one, solution to be

$$1+3m\cos\psi<0.$$

§ 6. *Condition for two Solutions.*—Consider first

$$0°<\psi<\tan^{-1}2,$$

so that the critical point lies to the left of both inflexions, as in fig. 1. Then we see that from C we can draw two tangents to the curve. The first has its point of contact, T_1, between V and M: this tangent, therefore, makes an acute angle with the horizontal axis and corresponds to a negative value, $-m_1$, of m, which is less in absolute value than $\frac{1}{3c}$. The second point of contact T_2 lies beyond W: this tangent has therefore a positive m, say m_2.

Now inspection shows that from $m=-\frac{1}{3c}$ to $m=-m_1$ the

line (5), falling between the tangents at C and T_1, cuts the curve twice to the right of C, the points of intersection falling one to the right, the other to the left of T_1. Hence for such values of m as lie between $-\frac{1}{3c}$ and $-m_1$ there are two solutions.

From $m=-m_1$ to $m=m_2$ the line falls between the tangents at T_1 and T_2, and does not, therefore, cut the curve at all except at the critical point.

For $m=m_2$ to $m=+\infty$, on the other hand, the line obviously cuts the curve twice to the right of the axis of symmetry, and, therefore, there are again two solutions.

Next let

$$\tan^{-1} 2 < \psi < 90°;$$

the critical point C then lies between V and M, as in fig. 2. We see here that the tangent at T_1 has no longer any interest, since the two intersections lie now to the *left* of C.

Thus from $m=-\frac{1}{3c}$ to $m=m_2$ there are no solutions, and from $m=m_2$ to $m=+\infty$ two solutions.

Next let

$$90° < \psi < \tan^{-1}(-2), \ (i.e.\ 116° \ 33' \ 52'');$$

the critical point C then lies between M and W, but the fig. 2 may otherwise serve our purpose. There will then evidently be no solutions from $m=-\infty$ to $m=m_2$, and two solutions from $m=m_2$ to $m=\frac{1}{-3c}$.

Lastly if

$$\tan^{-1}(-2) < \psi < 180°,$$

the critical point lies to the right of both inflexions, and two solutions are not possible for any value of m.

§ 7. *Analytical Criteria for Existence of two Roots.*

From the equation (6) we see that we have m_1 or m_2 according as we substitute the coordinates of T_1 or T_2 in the expression $\frac{\lambda^5}{3x}$.

To find the coordinates of T_1 or T_2 we write down the equation of the tangent at xy, and make it pass through the critical point $(-c, 1)$.

$$\lambda^5 (y-1) + 3x(x+c) = 0;$$

or, using (4) and substituting for λ^2 its value,

$$\lambda^5 = 4x^2 + 3cx + s^2.$$

Rationalising this we have to determine x the equation

$$(x^2+s^2)^5 = (4x^2+3cx+s^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

One root of this equation is, of course, $x=-c$, corresponding to the tangent at the critical point itself. This root is a double root, and we might, if we chose, divide the whole equation by $(x+c)^2$; this is, however, quite unnecessary, as this root can cause no confusion.

The point of contact T_2 corresponds to the positive root of the equation. When $\psi=116^\circ\ 33'\ 52''$ this root is $\frac{s}{2}$ or $\frac{1}{\sqrt{5}}$ and gradually increases as ψ diminishes, till when $\psi=0$, the root is very nearly 1·8. It may quite easily be calculated from the equation by Horner's method.

The point of contact T_1 corresponds to a negative root which has only to be calculated when $0^\circ<\psi<63^\circ\ 26'\ 8''$, and which is zero when $\psi=0$, and gradually increases up to $\frac{1}{\sqrt{5}}$. In calculating this root for small values of ψ we have to be careful, as there is another root of the equation somewhat greater in absolute magnitude than the one we require, which is also zero when $\psi=0$. This root, however, becomes imaginary for a value of ψ less than $63^\circ\ 26'\ 8''$,* so that with care there ought to be no confusion between the two.

In this way a table may be calculated giving for successive values of ψ the limits within which m must lie in order that there may be one, two, or no solutions of the problem. As follows :—

ψ	m			
	One solution.	Two.	None.	Two.
0 to $63^\circ\ 26'\ 8''$	$-\infty$ to $-\frac{1}{3c}$	$-\frac{1}{3c}$ to $-m_1$	$-m_1$ to m_2	m_2 to $+\infty$
$63^\circ\ 26'\ 8''$ to 90°	$-\infty$ to $-\frac{1}{3c}$		$-\frac{1}{3c}$ to m_2	m_2 to $+\infty$
90° to $116^\circ\ 33'\ 52''$	$\frac{1}{-3c}$ to ∞		$-\infty$ to m_2	m_2 to $\frac{1}{-3c}$
$116^\circ\ 33'\ 52''$ to 180°	$\frac{1}{-3c}$ to ∞		$-\infty$ to $\frac{1}{-3c}$	

The actual numbers are printed in Oppolzer's table XIII., b.

§ 8. *Limits of the Solution* $z\ (=x+c)$.

z being the distance of the astronomical point from the critical line, it is at once evident that, when m is positive, z is less than m. For in this case the astronomical point must lie between the critical point and the intersection of the line (5) with the horizontal axis. When, however, m is negative, it is evident that the

* When the tangent at the left-hand lower inflexion passes through the critical point.

astronomical point must lie in that part of the curve which lies between the critical point C and its reflexion K in the axis of symmetry. Hence when m is negative $z < 2$. This inequality is, I believe, new, though it is perfectly obvious from Lambert's curvature theorem without reference to the curve. I give it here chiefly because Oppolzer makes a misstatement on this point when he says that, m being negative, z may have any value from 0 to ∞. This mistake has not, however, crept into his tables.

§ 9. *Note on Lambert's Curvature Theorem.*

Let us imagine the Sun to be at the origin of coordinates O. Now when m is negative, the astronomical point must lie in that part of the curve which lies between the critical point C and its reflexion K in the axis of symmetry, *i.e.* in that part of the curve which is concave toward the Sun, like the apparent path of the comet in this case.

Similarly when m is positive the astronomical point must fall beyond the two points C, K, and therefore in the part of the curve which is convex toward the Sun like the apparent path of the comet in this case. This serves as an easy *memoria technica* for the relation between the sign of m and the concavity or convexity of the apparent path.

Further, since between C and K λ is less than unity, being equal to unity at C and at K, and greater than unity beyond C and K, it follows that when m is positive the astronomical point has its λ greater than unity, and when m is negative less than unity.

Remembering that for the astronomical point λ is the ratio of the heliocentric distances of the comet and Earth, this gives us at once Lambert's curvature theorem, viz. *the comet is nearer the Sun than the Earth is if the apparent path be concave toward the Sun, but is further away than the Earth if it be convex.* The figure is again a great help to the memory.

§ 10. *Note on Geometrical Discussions of the Problem.*

In Gauss's *Theoria Motus*, edited by C. H. Davis, Boston, 1857, references will be found, and abstracts of all the papers on the subject which had appeared up to that date. Of these Encke's original paper on the subject (1848) appears to be the fullest. The curves used by him are, however, complicated, since he treats the equation in the form

$$m \sin 4z = \sin (z - q).$$

Several methods by means of a straight line and curve are given in the book referred to. In all of them, however, the construction of the straight line is made to depend on both

parameters (m, ψ), while the curve is fixed independently. That which has most resemblance to my method is M. Binet's in the *Journal de l'École Polytechnique*, 20 cahier, tome xiii. p. 285. He uses the fixed curve

$$y = \frac{1}{(1+x^2)^{\frac{3}{2}}}.$$

The discussion is, however, very short, and aims only at showing that more than two solutions are impossible, and at obtaining the criterion for a single root. Many years later (1882) Oppolzer used the curve

$$y = \frac{z}{\{1 - 2z \cos \psi + z^2\}^{\frac{3}{2}}}$$

to prove only that more than two solutions are impossible. Here the straight line being inclined at 135° to the axis of z is fixed in direction, and therefore easy to construct, but the discussion is made difficult by the presence of both parameters in the equation to the curve. Oppolzer indeed not only dismisses the geometry very shortly, but gives in his book no general theory, or even an explanation of the mode in which his tables were constructed.

In my own treatment of the subject I have aimed at keeping the two parameters (m, ψ) geometrically distinct, so as to simplify the theory and the classification of the different cases.

Cambridge: March 2, 1897.

On the Orbit of the November Meteors. By Prof. J. C. Adams.*

It is known to the President and to several members of the Society that I have been for some time past engaged in researches respecting the November Meteors, and allusion is made to some of my earlier results in the last Annual Report. As my investigations are now in some measure complete, and the results which I have obtained appear to me important, I have thought that they may not be without interest for the Society.

In a memoir on the November Star Showers, by Professor H. A. Newton, contained in Nos. 111 and 112 of the *American Journal of Science and Arts*, the author has collected and discussed the original accounts of 13 displays of the above phenomenon in years ranging from A.D. 902 to 1833.

The following table exhibits the dates of these displays, and the Earth's longitude at each date, together with the same particulars for the shower of November last, which have been added for the sake of completeness.

* This paper, by the late Professor Adams, originally appeared in *Monthly Notices*, vol. xxvii. p. 247 (1867 April), but there are now not many copies of the number left in stock. Applications for copies have already been made to the Council, and in view of the return of the *Leonids* in the next few years they have determined to reprint the paper in the current number.—Eds. *M.N.*